Indian Statistical Institute
Mid-Semestral Examination
Differential Geometry II - BMath III

Max Marks: 40 Time: 180 minutes.

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

(1) Decide whether the following statements are TRUE or FALSE. Justify.
(a) Given a smooth manifold M, $p \in M$ and $v \in T_p(M)$, there exists $X \in \Gamma(M)$ with $X_p = v$.
(b) Every $1-1$ immersion is an embedding.
(c) Every smooth map $f : \mathbb{RP}^2 \longrightarrow \mathbb{R}^2$ has a critical point.
(d) Every bijective submersion is a diffeomorphism.
(e) The set of critical values of any smooth map $f : S^2 \longrightarrow \mathbb{R}$ is closed.
(f) No proper subset of S^2 is diffeomorphic to S^2. [$2 \times 6 = 12$]

(2) (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (non constant) smooth map with support contained in $(-1/2, 1/2)$. List the rationals in a sequence: $q_1, q_2, \ldots$. Consider the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x) = \sum_n q_n f(x-n).$$

Show that g is smooth. Further, if C denotes the set of critical points of g, show that $g(C)$ is dense in $\mathbb{R}$. [$2+2=4$]
(b) Show that $O(n) = \{A \in M_n(\mathbb{R}) : AA^t = I\}$ is a smooth manifold. Determine its dimension. For $A \in O(n)$ describe the tangent space to $O(n)$ at A. [$2+1+2$]
(c) Let $f : M \longrightarrow N$ be a surjective submersion. Show that for each $p \in M$ there is a neighbourhood U of $f(p)$ and a smooth map $g : U \longrightarrow M$ such that $f \circ g = id$. [3]

(3) (a) Give an example each of a vector field X on S^1 having exactly one zero and a vector field Y on S^2 having exactly two zeros. Give complete proofs of your assertions. [$2+2=4$]
(b) Decide whether the following vector fields are complete. (a) $X = x^2(d/dx)$ on $\mathbb{R}$. (b) $Y = y^2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ and (c) $Z = \sum r_i \frac{\partial}{\partial r_i}$ on $\mathbb{R}^n$. Justify your answers. [2+2+2=6]
(c) A critical point p of a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be non-degenerate if the $n \times n$ matrix

$$\left(\frac{\partial^2 f}{\partial r_i \partial r_j}\right)$$

is invertible (at p). Show that 0 is a non-degenerate critical point of $\det : M_n(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ when $n = 2$ and not a non-degenerate critical point when $n > 2$. [3+3=6]